THE SPHERICAL BERNSTEIN PROBLEM
IN EVEN DIMENSIONS AND RELATED PROBLEMS

by

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0. Introduction

The minimal surface equation is probably the best known among the non-linear, elliptic partial differential equations, and has been studied extensively. In Euclidean space R^3 the classical Bernstein theorem states that any solution which is an entire minimal graph over R^2, must be a plane. In a celebrated sequence of investigations the combined efforts of de Giorgi (8), Almgren (1), Simons (16), Bombieri, de Giorgi and Giusti (3) succeeded in extending this result to R^n, $n<8$, and providing counterexamples for $n>8$. At the 1970 International Congress of Mathematicians in Nice, Professor S.S. Chern proposed the following as one outstanding problem in differential geometry:

The Spherical Bernstein Problem: Let the $(n-1)$-sphere be imbedded as a minimal hypersphere in the standard Euclidean n-sphere $S^n(1)$. Is it necessarily an equator?

For $n = 3$ the answer to the above problem was already known to be positive by a theorem of Almgren and Calabi, which holds under the weaker assumption of an immersed S^2 in $S^3(1)$. No further progress was made until Wu-Yi Hsiang recently proved the existence of infinitely many non-congruent minimal imbeddings of S^{n-1} into $S^n(1)$ for the specific dimensions $n = 4,5,6,7,8,10,12,14$. (9,10).

In this paper we solve the spherical Bernstein problem simultaneously for all even n.

Theorem

Let $S^{2m}(1)$ be the standard Euclidean sphere of dimension $2m$. Then there exists a minimally imbedded $(2m-1)$-sphere which is different from the equator.

The proof given here suggests that there in general should be only one almost-homogeneous example, invariant under the isometry group $SO(2)\times SO(m)$ on $S^{2m}(1)$ defined below; i.e. a remarkable "second-best equator". Similarly the same type of construction gives an almost-homogeneous $S(U(2)\times U(m))$-invariant example on $S^{4m}(1)$ and an $Sp(2)\times Sp(m)$-invariant example on $S^{8m}(1)$.

There are notable differerences between the examples constructed here and those of Wu-Yi Hsiang referred to above. In Hsiang's construction essential use was made of some unstable minimal cones of focal type (related to the local geometry of the corner singularity of the orbit space of $S^{2m}(1)$). The oscillatory behaviour of a dynamical system near a singularity of focal type in that case eventually produces infinite families of minimal hyperspheres. The constructions of this paper shows that in addition to those infinite families, which occur only for a few low dimensions, there exist examples of minimal hyperspheres of generalized rotational type whose construction is based on areaminimizing homogeneous cones, corresponding to a corner singularity of nodal point type. The difficulties in this case has up to now been a major obstacle to extending constructions of minimal hyperspheres to larger classes of symmetric spaces, (see (13) for extensions to other symmetric spaces in the focal point case).

The Spherical Bernstein Problem has a direct bearing on the problem of the local structure of an isolated singularity p of a minimal hypersurface N^n of a Riemannian manifold M^{n+1}. The tangent cone of N^n at p is a minimal cone in R^{n+1}, whose intersection Q with $S^n(1)$ is a minimal hypersurface. Hence the theorem of Almgren-Calabi shows that for $n = 3$, Q cannot be a sphere, (i.e. N^n a topological manifold) unless N^n is smooth at p ; i.e. the theorem is analogous to Mumfords theorem for isolated singularities of complex, algebraic surfaces.

On the other hand, the cone construction (with vertex at the origin) on our minimal hyperspheres, gives the following theorem, which demonstrates an analogous role in the theory of isolated singularities of minimal hypersurfaces of Riemannian manifolds as that of the spheres of Brieskorn (4) for isolated singularities of complex hypersurfaces in algebraic geometry.

Theorem.

An isolated singularity of a minimal hypersurface of an odd-dimensional Euclidean space R^{2m+1} cannot in general be detected by its local topological structure.

In view of the well-known difficulties in finding closed solutions, even in the case of ordinary differential equations, it is not surprising that a considerable amount of explicit non-linear analysis is required. Our result is an existence theorem, our minimal hyperspheres are non-homogeneous and not given by any explicit equation. Thus, from the point of view of geometric measure theory, their analysis is more complicated than most examples studied in depth earlier, and could involve computer assisted approximations. The construction suggests stronger stability properties for the cones over these examples than in the focal point case*.

For further observatons on minimal cones and the Spherical Bernstein problem, see (10).

Our construction is based on the orbital geometry of the transformation group $G = SO(2)\times SO(m)$ acting on $S^{2m}(1) \subseteq R^{2m}\oplus R = R^2\otimes R^m\oplus R$ by the representation $\rho_2\otimes\rho_m\oplus 1$ (here ρ_k is the standard representation of $SO(k)$ on R^k and 1 is the trivial representation). We can then apply methods of equivariant differential geometry; this approach was initiated by Hsiang and Lawson (12), and has recently been applied by Hsiang to obtain some strong results (9, 10); we would like to acknowledge our debt to his work. We present in this article a careful exposition of the relevant methods from equivariant differential geometry at their present stage of refinement.

In section 1 we study the orbit map from $S^{2m}(1)$ to $S^{2m}(1)/G$. The restriction of this to the generic set of principal orbits is a Riemannian submersion in the sense of O'Neill (15). The calculation of the mean curvature of a hypersurface requires only the following data: the orbital distance metric on the orbit space (a spherical lune) and the volume functional, which registers the volume of the fibres. Since the representation $\rho_2\otimes\rho_m$ of G is the isotropy representation of the Grassmannian manifold of oriented 2-planes in R^{m+2}, this is essentially an application of

* Very recently, Hsiang and Sterling have shown that the cones over many of the minimal hyperspheres of our main theorem, are indeed stable.

the theory of Élie Cartan and Hermann Weyl to a specific case. With these results, we deduce in section 2 the differential equation in orbit space for a G-invariant minimal hypersurface of $S^{2m}(1)$. Our investigation is then reduced to finding special types of solution curves. This requires a considerable amount of non-linear analysis in orbit space, both qualitative arguments and specific estimates. In section 3 the differential equation is studied at the singular boundary. In section 4 the equation is deformed to a homothetically invariant diferential equation, which at the corner singularity is a good local approximation. The latter equation is then analyzed by Poincaré-Bendixon theory. For completeness, we also include a sharpening of a proposition of Lawson, which produces examples of area-minimizing homogeneous cones, i.e. non-interior regularity of solutions to the Plateau problem (14).

Section 5 contains some qualitative analysis of solution curves. In particular we establish a criterion for the existence of solution curves which oscillate between the two smooth arcs of the singular boundary; this is more generally applicable than previous methods (9, 10). In fact it can be applied to show the following theorem: Any standard sphere $S^n(1)$, $n>3$, has infinitely many non-congruent, minimally immersed hyperspheres.

Hence the theorem of Almgren and Calabi fails for all dimensions higher than 3. Our method is based on counting critical points along certain segments of solution curves.

The main analysis is carried out in section 6 and 7. It is a somewhat annoying feature of non-linear analysis that arguments tend to be unconvincing until specific numerical estimates have been made; for the benefit of the sceptical reader sufficient details are given in the Appendix. Our main theorem is finally established by studying the variation of the above mentioned number of critical points along a one-parameter family of solution curves emanating from the singular boundary of the spherical lune.

1. The orbital geometry.

Let G/K be a symmetric space of compact type, and let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. Let H be a fixed principal isotropy group of the isotropy representation of K on $\mathfrak{p}$. The fixed point set $\mathfrak{a} = F(H,p)$ is characterized as a maximal Abelian subspace of $\mathfrak{p}$. Let $M = \{k\in K;\ ad(k)/\mathfrak{a} = id\}$ and $M' = \{k\in K;\ Ad(k)(\mathfrak{a}) \subseteq \mathfrak{a}\}$, then the relative Weyl group is $W = M'/M$. Let $\mathfrak{g}_C = \mathfrak{k}_C+\mathfrak{p}_C$ be the complexifications, let $\mathfrak{h}_0$ be a maximal Abelian subalgebra of $\mathfrak{m}$, and let $\mathfrak{h} = \mathfrak{h}_0+\mathfrak{a}$; then $\mathfrak{h}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$. Let Δ be the root system of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$, then $\Sigma = \{\alpha|\mathfrak{a}\ ;\ \alpha\in\Delta\}$ is the restricted root system. W is generated by reflections in the hyperplanes of $\mathfrak{a}$ annihilated by restricted roots. The following results from Cartan-Weyl theory generalize standard facts in the special case of the adjoint representation of a compact Lie group:

Proposition 1.

The orbit space $\mathfrak{p}/K\simeq\mathfrak{a}/W$; i.e. the orbit space, with the orbital distance metric, can be identified with a Weyl chamber C in $\mathfrak{a}$. For an interior point x of C, the volume of the principal orbit $Ad(K)\cdot x$ is given by $v(x) = c \prod_{\alpha\in\Sigma^+}|i\alpha(x)|$, where c is a constant and Σ^+ a positive system of restricted roots.

We now specify to the example $G = SO(m+2)$, $K = SO(2)\times SO(m)$, $\mathfrak{k} = \mathfrak{so}(2)\times\mathfrak{so}(m)$, $\mathfrak{p} = \{\begin{pmatrix} 0 & X \\ -{}^tX & 0\end{pmatrix}, X$ a $(2\times m)$-matrix$\}$, (${}^tX$ is the transpose of X).

Let $\mathfrak{a} = \{\begin{pmatrix} 0 & X \\ -{}^tX & 0\end{pmatrix}, X = \begin{pmatrix} 0 & 0 & 0 & x_1 & 0\cdots 0 \\ 0 & 0 & x_2 & 0 & 0\cdots 0\end{pmatrix}\}$.

Let $\mathfrak{t}$ be the Lie algebra of the standard maximal torus of $\mathfrak{so}(m+2)$, i.e. $\mathfrak{t} = \left\{\begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}\right\}$ and let $\mathfrak{h}_0$ be the subspace of $\mathfrak{t}$ defined by $x_1 = x_2 = 0$. Then $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{a}$ is conjugate in $\mathfrak{so}(m+2)$ to $\mathfrak{t}$ (by a base change in the first four variables). It follows that the root system Δ of $\mathfrak{so}(m+2)$ w.r.t. $\mathfrak{h}$ is given by the standard formulas:

$$m = 2n: \quad i(x_r - x_s),\ \pm i(x_r + x_s) \quad \text{for} \quad r \neq s.$$

$$m = 2n+1: \quad \pm ix_r,\quad i(x_r - x_s),\quad \pm i(x_r + x_s) \quad \text{for} \quad r \neq s.$$

The restricted root system $\Sigma = \Delta|\mathfrak{a}$ is defined by $x_3 = x_4 = \cdots x_{n+1} = 0$, i.e. Σ consists of $\pm i(x_1 - x_2)$, $\pm i(x_1 + x_2)$ with multiplicity 1, and $\pm ix_1$, $\pm ix_2$ with multiplicity $m-2$. A system of simple roots for Σ can be chosen as $\{ix_2, i(x_1 - x_2)\}$, then $i(x_1 + x_2)$ becomes the highest restricted root. In the (x_1, x_2)-plane the fundamental domain (i.e. the orbit space $\mathfrak{p}/K$) is the Weyl chamber $x_1 \geq 0$, $x_1 \geq x_2$.

The isotropy representation of $SO(2)\times SO(m)$ on $\mathfrak{p}$ is easily identified as the representation $\rho_2 \otimes \rho_m$ on R^{2m}. Hence the orbit space of $\rho_2 \otimes \rho_m \oplus 1$ on R^{2m+1} is the domain $x_2 \geq 0$, $x_1 \geq x_2$ in (x^1, x^2, x^3)-space. Restricting to $S^{2m} \subseteq R^{2m+1}$, the orbit space is $X = \{(x^1, x^2, x^3) | x_2 \geq 0,\ x_1 \geq x_2,\ x_1^2 + x_2^2 + x_3^2 = 1\}$, i.e. a spherial lune of angle $\frac{\pi}{4}$. From proposition 1 the volume functional is $v(x_1,x_2,x_3) = cx_1^{m-2}x_2^{m-2}(x_1-x_2)(x_1+x_2)$. Introducing spherical polar coordinates (r,θ) centered at the north pole in X, we obtain:

The orbital distance metric on X is: $ds^2 = dr^2+\sin^2 r d\theta^2$.

The (normalized) volume functional is:

$v(r,\theta) = \sin^{2m-2} r \sin^{m-2} 2\theta \cos 2\theta$.

2. The reduced minimal equation in orbit space.

Let M be a Riemannian manifold with a compact isometry group G of cohomogeneity 2 (i.e. principal orbits have codimension 2). A minimal hypersurface N of M is characterized by a system of non-linear partial differential equations of elliptic type. Let $\pi: M \to X = M/G$ be the orbit projection, let M^* be the union of principal orbits; the restriction $\pi/M^*: M^* \to X^* = M^*/G$ is then a Riemannian submersion in the sense of O'Neill (15). If N is a G-invariant hypersurface, the computation of the mean curvature of N has a particularly simple reduction in terms of the geometry of the orbit space and the fibres, enabling us to reduce the above system of partial differential equations to a non-linear differential equation in orbit space.

Proposition 2.

Let M, G, X, π, M^*, X^* be as above. Let γ be a curve in X^*, and let $N = \pi^{-1}(\gamma)$ be its inverse image under π in M. Then we have: $H(z) = k(\pi(z)) - \frac{d}{d\bar{n}} \ln v(\pi(z))$ for $z \in N$. Here H is the mean curvature of N, k is the geodesic curvature of γ, and $\bar{n}$ is the oriented normal of γ.

Remark. This result from equivariant differential geometry has been applied in several recent papers (9, 10). It is easily demonstrated by applying the first variation formula for the volume of N: $v'(0) = -\int_N \langle\phi\bar{K},\bar{H}\rangle$ (the boundary term vanishes) to compactly supported, equivariant variations with normal vector field $\phi\bar{K}$, and observing that the volume of N is given by $\int_\gamma v\,ds$. (v is the volume functional on X^* as in section 1, ds is the orbital distance metric, and $\bar{H}$ is the mean curvature vector, compare (12).)

From now on we speify to $G = SO(2)\times SO(m)$ acting on $M = S^{2m}(1) \subseteq R^{2m+1}$ by $\rho_2\otimes\rho_m\oplus 1$. From section 1:

X is the spherical lune parametrized by $(r,\theta)\in[0,\pi]\times[0,\frac{\pi}{4}]$

$X^* = \text{int } X.$ $ds^2 = dr^2+\sin^2 r\,d\theta^2.$ $v(r,\theta) = \sin^{2m-2} r\, \sin^{m-2} 2\theta\cos 2\theta.$

Theorem 1.

Let N be a compact, G-invariant hypersurface of M. Let $\gamma(s)$ be the curve $\pi(N)$ in X parametrized by arc length s, and let α be the angle from $\frac{\partial}{\partial r}$ to the tangent $\frac{d\gamma}{ds}$. Then N is minimal if and only if the generating curve $\gamma \cap X^*$ satisfies the following differential equation:

$$(*)\quad \begin{aligned} \dot{r} &= \cos\alpha \\ \dot{\theta} &= \sin\alpha \sin^{-1} r \\ \dot{\alpha} &= -(2m-1)\sin\alpha \sin^{-1} r \cos r + 2\cos\alpha \sin^{-1} r[(m-2)\cot 2\theta - \tan 2\theta]. \end{aligned}$$

Proof: We observe that there are no exceptional orbits. By a well-known dimension argument in transformation groups, N must intersect M^*, hence $(N \cap M^*)$ is open dense in N. N is minimal

if and ony if $H \equiv 0$, by continuity it suffices to check this on $N \cap M^*$. By Proposition 2 this reduces to $k(\gamma(s)) - \frac{d}{d\bar{n}} \ln v(\gamma(s)) = 0$ on $\gamma \cap X^*$. Here $\dot{\gamma} = \dot{r}\frac{\partial}{\partial r} + \dot{\theta}\frac{\partial}{\partial \theta}$ and $\|\frac{\partial}{\partial \theta}\| = \sin r$. From $\cos\alpha = \langle\frac{\partial}{\partial r}, \dot{\gamma}\rangle$ and $\sin\alpha = \cos(\frac{\pi}{2} - \alpha) = \langle\dot{\gamma}, \sin^{-1} r \frac{\partial}{\partial \theta}\rangle = \dot{\theta}\sin^{-1} r \|\frac{\partial}{\partial \theta}\|^2$ we deduce the first two equation of $(*)$. With orientation defined by the coordinate system (r,θ), we have: $\bar{n} = -\dot{\theta}\sin r \frac{\partial}{\partial r} + \dot{r}\sin^{-1} r \frac{\partial}{\partial \theta}$.
$\frac{d}{d\bar{n}}(\ln v) = \frac{d}{d\bar{n}}(2(m-2)\ln\sin r + (m-2)\ln\sin 2\theta + \ln\cos 2\theta] =$
$-2(m-1)(\cos r)\dot{\theta} + (m-2)2\sin^{-1} r(\cot 2\theta)\dot{r} - 2\sin^{-1} r(\tan 2\theta)\dot{r}$. From Liouville's formula applied to $\gamma(s) = (r(s),\theta(s))$ (see (6), p. 252), we obtain $k(\gamma(s)) = \dot{\alpha} + (\cos r)\dot{\theta}$, hence
$H = \dot{\alpha} + (\cos r)\dot{\theta} + 2(m-1)(\cos r)\dot{\theta} - (m-2)2\sin^{-1} r(\cot 2\theta)\dot{r} + 2\sin^{-1} r(\tan 2\theta)\dot{r}$.
Substitution of the first two equations of $(*)$ shows that $H = 0$ if and only if the third equation holds.

q.e.d.

Remark 1.

The equation $(*)$ is reflectionally symmetric around $r \equiv \frac{\pi}{2}$. It is also symmetric under reversal of parameter, i.e. if $\gamma(s) = (r(s),\theta(s))$ is a solution, then $\mu(s) = \gamma(-s)$ is also a solution.

Remark 2.

There are two easy solution of $(*)$:

(i) $r \equiv \frac{\pi}{2}$ is the equator $S^{2m-1}(1) = R^{2m} \cap S^{2m}(1)$.

(ii) $\theta \equiv \theta_0 = \frac{1}{2}\mathrm{Arctan}\sqrt{m-2}$ is the suspension of the principal orbit of maximal volume. It does not define a smooth submanifold of $S^{2m}(1)$ ("the meridian solution").

We now conclude with our main reduction theorem for minimal hyperspheres:

Theorem 2.

Let G, M, X, X^*, π be as above. Let $\gamma(s) = (r(s),\theta(s))$, $s\in(a,b)$ be a simple smooth curve in X^*, parametrized by arc length, such that $r(a+)\in(0,\pi)$, $\theta(a+) = 0$, $r(b-)\in(o,\pi)$, $\theta(b-) = \frac{\pi}{4}$. Assume that $(r(s),\theta(s),\ \alpha(s))$ is a solution of $(\star)$ for $s\in(a,b)$ with $\alpha(a+)\equiv\alpha(b-)\equiv\frac{\pi}{2} \bmod(2\pi)$. Then $N = \pi^{-1}(\gamma)$ is a minimally imbedded hypersphere of $M = S^{2m}(1)$.

Proof. The coordinate curve $r = c\in(0,\pi)$ generates the hypersphere $\{(x,z)\,|\,z = \cos c,\ \|x\|^2+z^2\equiv1\}$ in $M = S^{2m}(1)$. A curve in X that enters the boundary $\theta\equiv0$ (or $\theta\equiv\frac{\pi}{4}$) orthogonally, generates a smooth hypersurface in M, so $N = \pi^{-1}(\gamma)$ is a smooth, minimal hypersurface of M by Theorem 1. To conclude that N is a sphere, we note that is is of cohomegeneity 1 under G with $[a,b]$ as orbit space. Here (G_c) is the principal orbit type for $c\in(a,b)$, and the one-parameter family of orbit types (G_c), $c\in[a,b]$ corresponds exactly to the same data for the G-space $S^{2m-1}(1)$ generated by $r\equiv\frac{\pi}{2}$. It is well known from transformation group theory that those data determine N as the union of the mapping cylinders of the projections $G/G_0\to G/G_a$ and $G/G_0\to G/G_b$. Hence N must also be a $(2m-1)$-sphere.

q.e.d.

3. The differential equation at the singular boundary.

We will frequently need the following observations on solution curves of $(*)$.

Lemma 1.

Let $(r(s),\theta(s),\alpha(s))$ be a solution curve of $(*)$. We then have:

(i) any relative maximum (minimum) of $r(s)$ occurs with $r>\frac{\pi}{2}$ $(r<\frac{\pi}{2})$.

(ii) any relative maximum (minimum) of $\theta(s)$ occurs with $\theta>\theta_0$ $(\theta<\theta_0)$.

(iii) any relative maximum (minimum) of $\alpha(s)$ occurs with α in the first or third (second or fourth) quadrant.

Proof. From $(*)$ we have: $\ddot{r} = -\dot{\alpha}\sin\alpha$, at $\dot{r} = \cos\alpha = 0$ we then have $\ddot{r} = (2m-1)\cot r$, and (i) follows. Similarly $\ddot{\theta} = \sin^{-2}r(\dot{\alpha}(\cos\alpha\sin r - \sin\alpha\cos\alpha\cos r)$, at $\dot{\theta} = 0$ we have $\ddot{\theta} = 2\sin^{-2}r\cos^2\alpha((m-2)\cot 2\theta - \tan 2\theta)$, and (ii) follows. Computing $\ddot{\alpha}$ and substituting the relation between α,θ,r defined by $\dot{\alpha} = 0$ yields $\ddot{\alpha} = K(r,\theta)\sin\alpha\cos\alpha$, where $K(r,\theta) = 2m-1-4\sin^{-2}r((m-2)\sin^{-2}2\theta+\cos^{-2}2\theta)$ is always negative.

q.e.d.

Proposition 3.

Let $\gamma(s) = (r(s),\theta(s))$, $s\in(-\varepsilon,\varepsilon)$ be a continuous curve in X, with $r(0)\in(0,\pi)$, $\theta(0) = 0(\frac{\pi}{4})$, and assume that $\alpha(s)$ is a differentiable function on $(-\varepsilon,0)\cup(0,\varepsilon)$ such that $(r(s),\theta(s),\alpha(s))$ satisfies $(*)$. Then $\alpha(0-) = -\frac{\pi}{2}$ and $\alpha(0+) = \frac{\pi}{2}$ (respectively $\alpha(0-) = \frac{\pi}{2}$ and $\alpha(0+) = -\frac{\pi}{2}$ if $\theta(0) = \frac{\pi}{4}$).

Proof. By reflectional symmetry we may assume $r(0) \leq \frac{\pi}{2}$. From $(*)$ it follows that as $\theta(s) \to 0$, the term $(m-2)\cot 2\theta$ dominates over $\tan 2\theta$, hence $\dot{\alpha} > 0$ at $\alpha = 0$ and $\dot{\alpha} < 0$ at $\alpha = -\pi$. It follows that there exists a $\delta > 0$ such that $\alpha(s) \in (-\pi, 0)$ for $s \in [-\delta, 0)$, otherwise $\alpha(s)$ would have to remain in $[0, \pi]$, i.e. $\dot{\theta} \geq 0$, contradicting $\theta(0-) = 0$. So $\alpha(s)$ has an accumulation point $\alpha_0 \in [-\pi, 0]$ as $s \to 0-$. Assume $\alpha_0 \in (-\frac{\pi}{2}, 0]$. Then $\frac{d\alpha}{d\theta} = \frac{\dot{\alpha}}{\dot{\theta}} = -(2m-1)\cos r + 2\cot\alpha[(m-2)\cot 2\theta - \tan 2\theta]$. Let $s_j \to 0-$, with $\alpha(s_j) \to \alpha_0$. Then $\frac{d\alpha}{d\theta}(s_j)$ is eventually dominated by the negative term $2(m-2)\cot\alpha \cot 2\theta$, and $\dot{\alpha}(s_j)$ is eventually positive. By Lemma 1, (iii), $\alpha(s)$ cannot have any relative maximum with $\alpha \in (-\frac{\pi}{2}, 0)$, hence $\dot{\alpha}(s)$ is eventually positive as $s \to 0-$. We then have the estimate: $\frac{d\alpha}{d\theta} < k \cot\alpha \cot 2\theta < 0$ for a positive constant k. The solution of the separable equation $\frac{d\alpha}{d\theta} = k \cot\alpha \cot 2\theta$ is $\cos\alpha = c(\sin 2\theta)^{-\frac{k}{2}}$. Hence $\cos\alpha > C(\sin 2\theta)^{-\frac{k}{2}} \underset{s \to 0-}{\to} \infty$, which is a contradiction. A similar argument yields a contradiction for $\alpha_0 \in [-\pi, -\frac{\pi}{2})$, so $\alpha_0 = -\frac{\pi}{2}$ is the unique accumulation point, and hence the limit of $\alpha(s)$ as $s \to 0-$. A similar argument gives $\alpha(0+) = \frac{\pi}{2}$, the same type of argument also works for the case $\theta(0) = \frac{\pi}{4}$.

q.e.d.

Corollary 1.

Let $\gamma(s) = (r(s), \theta(s))$, $s \in (-\varepsilon, 0]$, be a continuous curve in X with $r(0) \in (0, \pi)$, $\theta(0) = 0$ or $\frac{\pi}{4}$, which defines a solution curve of $(*)$ for $s \in (-\varepsilon, 0)$. Then γ is analytic.

Proof. By Theorem 1 and the proof of Theorem 2, the lift $\pi^{-1}(\gamma)$ is a smooth, minimal hypersurface of $S^{2m}(1)$. Analyticity follows by standard regularity theorems.

q.e.d.

The differential equation (*) becames singular at the boundary of X ($\theta = 0$ or $\theta = \frac{\pi}{4}$). The question of the existence and uniqueness of analytic solution curves originating at the boundary was investigated (for a special case) in (11) by a method which is generally applicable. We indicate how the present case is reduced to the treatment in (11).

Theorem 3.

Let $(b,0)$ with $b\in(0,\pi)$ be a point on the singular boundary of X. Then there exists a unique, continuous curve $\gamma_b(s) = (r(s),\theta(s))$ in X, which defines a solution $\Gamma_b(s) = (r(s),\theta(s),\alpha(s))$ of (*) for $s>0$, with $r(0) = b$, $\theta(0) = 0$. Here $\Gamma_b(s)$ is analytic in (b,s) (as long as s is restricted to an interval of the form $[0,c]$ where the curve $\gamma_b(s)$ does not intersect the singular boundary again), and $\alpha(0+) = \frac{\pi}{2}$.

Proof. By Corollary 1 it suffices to demonstrate the existence of an analytic solution and its uniqueness within the set of analytic solutions. We perform a change of variable from s to θ, and consider $r = r(\theta)$. Then $\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \sin r \cot \alpha$. Let $p = \sin r \cot \alpha$. Then $\frac{dr}{d\theta} = p$ and $\theta \frac{dp}{d\theta} = \theta(p \cos r \cot \alpha - \dot{\alpha} \sin^2 r \sin^{-3}\alpha)$. Substitution from (*) and computation gives: $\theta \frac{dp}{d\theta} = \theta p^2 \cot r + (2m-1)\theta \sin r \cos r(1+p^2\sin^{-2} r) - 2(m-2)p(1+p^2\sin^{-2} r)\theta \cot 2\theta + 2p(1+p^2\sin^{-2} r)\theta \tan 2\theta$, where we are interested in solutions

$r(t,\theta)$ with initial conditions $r(t,0) = b+t$, $\frac{dr}{d\theta}(t,0) = 0$. With $\bar{r}(t,\theta) = r(t,\theta)-t-b$, and expansion of the terms in the above expression we then obtain:

$$\frac{d\bar{r}}{d\theta} = p$$

$$\theta \frac{dp}{d\theta} = \lambda p + a_{0100}\theta + \sum_{\substack{1+q+n+\nu \geq 2 \\ q+n+\nu \geq 1}} a_{1qn\nu}(b) t^{1} \theta^{q} \bar{r}^{n} p^{\nu},$$

with $\lambda = -(m-2)$, $a_{0100} = (2m-1)\sin b \cos b$.

This is precisely of the form considered in (11), so formal power series substitution and majorization gives uniqueness and existence of an analytic solution $\bar{r}(t,\theta)$ in a neighbourhood of 0 with $\bar{r}(t,0) = 0$, $\frac{d\bar{r}}{d\theta}(t,0) = 0$; i.e. $r(t,0) = b+t$, $\frac{dr}{d\theta}(t,0) = 0$, as desired. Combination with standard analytic dependance on initial conditions in a regular region concludes the proof of Theorem 3.

q.e.d.

Corollary 2.
Let $\gamma(s) = (r(s),\theta(s))$ and $(r(s),\theta(s),\alpha(s))$ be as in Proposition 3. Then $(r(s),\theta(s),\alpha(s)) = (r(-s),\theta(-s),\alpha(-s)+\pi)$ for $s \in (0,\varepsilon)$.

Proof. Define $(r_1(s),\theta_1(s),\alpha_1(s)) = (r(-s),\theta(-s),\alpha(-s)+\pi)$ for $s \in [0,\varepsilon)$. By Remark 1 this is a solution of $(*)$ for $s \in (0,\varepsilon)$, by Proposition 3 and the uniqueness result of Theorem 3 it must coincide with $(r(s),\theta(s),\alpha(s))$ for $s \in [0,\varepsilon)$.

Hence any solution curve which hits the singular boundary continues back along the same trajectory, with a discontinuous jump in α at the boundary. Closeby solution curves will generically avoid the boundary, i.e. $\alpha(s)$ is smooth; by the next proposition $\alpha(s)$ will nevertheless turn sharply near the boundary.

Definition 1.

Let $\Gamma_{r,\theta,\alpha}(s) = (r,(s),\theta(s),\alpha(s))$ be the unique solution curve of $(*)$ with initial conditions $r(0) = r$, $\theta(0) = \theta$, $\alpha(0) = \alpha$, where $r\in(0,\pi)$, $\theta\in(0,\frac{\pi}{4})$, $\alpha\in R(\text{mod } 2\pi)$, and let $\gamma_{r,\theta,\alpha}(s) = (r(s),\theta(s))$ be its projection to the orbit space. We extend to initial conditions on the lower boundary $\theta = 0$ by defining $\Gamma_r(s) = (r(s),\theta(s),\alpha(s))$ as the unique solution curve with $r(0) = r$, $\theta(0) = 0$, and $\gamma_r(s) = (r(s),\theta(s))$ its projection.

Proposition 4.

Let $b\in(0,\pi)$ and $\varepsilon\in(0,\frac{\pi}{8})$. Then there exists a positive δ such that for any θ in $(0,\delta)$ (resp. in $(\frac{\pi}{4}-\delta,\frac{\pi}{4})$ there exists an s_0 in $(0,\varepsilon)$ such that with $\Gamma_{b,\theta,\alpha}(s) = (r(s),\theta(s),\alpha(s))$ we have $\theta(s_0)\in(0,\varepsilon), |\frac{\pi}{2}-\alpha(s_0)|<\varepsilon$ (respectively $\theta(s_0)\in(\frac{\pi}{4}-\varepsilon,\frac{\pi}{4}), |\frac{\pi}{2}+\alpha(s_0)|<\varepsilon)$.

Proof. We may choose $\delta<\theta_0$ and $b<\frac{\pi}{2}$ by Remark 1.

(a) $\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}-\varepsilon)$. From $(*)$ it follows that $\dot{\alpha}(s)\geq 0$ as long as $\alpha(s)\in[-\frac{\pi}{2},0)$ and $r(s)<\frac{\pi}{2}$. For $r(s)>\frac{\pi}{2}$ it follows from Lemma 1, (iii) that $\alpha(s)$ cannot reach a relative maximum for $\alpha(s)\in[-\frac{\pi}{2},0)$, hence $\dot{\alpha}(s)\geq 0$ until $\alpha(s) = 0$, and $\Gamma_{b,\theta,\alpha}(s)$ cannot enter the boundary in this region, (except in the special case $b = \frac{\pi}{2}$, $\alpha = -\frac{\pi}{2}$; i.e. the equator solution). By $(*)$ we have $\dot{\theta}(s)\leq\sin\alpha(s)$

for $\alpha(s)\in(-\frac{\pi}{2},0)$, hence, by choosing δ sufficiently small, we obtain $\alpha(s)>-\varepsilon$ before $s=\frac{\varepsilon}{10}$. Let $K_\theta=(m-2)\cot 2\theta-\tan 2\theta$, then $K_\theta\to\infty$ as $\theta\to 0+$; furthermore, by choosing δ small enough, the term $2\cos\alpha\sin^{-1}r\ K_\theta$ dominates $\dot{\alpha}$ until $\alpha(s)=0$; so we can obtain $\alpha(s)=0$ for an $s=s_1\in(0,\frac{\varepsilon}{8})$. Now determine θ_1 such that $\dot{\alpha}>2\cos\alpha\sin^{-1}r\cot 2\theta$ for $\alpha\in[0,\frac{\pi}{2}-\varepsilon]$, $\theta\in(0,\theta_1)$; then $\frac{d\alpha}{d\theta}>2\cot\alpha\cot 2\theta$. Comparing with the equation $\frac{d\alpha}{d\theta}=2\cot\alpha\cot 2\theta$ with initial condition $\alpha(\theta_2)=0$, as in the proof of Proposition 3, we obtain $\cos\alpha(\theta)<\sin 2\theta_2\sin^{-1}2\theta$ for $0<\theta_2<\theta<\theta_1$. Choose n so large that $1<2^{n-1}\varepsilon$ and then θ_2 so small that $2^{n+2}\theta_2<\max(\theta_1,\frac{\varepsilon^2}{2})$ and $\cos^{n+1}(2^{n+1}\theta_2)>\frac{1}{2}$. Then $\cos\alpha(2^{n+1}\theta_2)\leq \sin 2\theta_2\sin^{-1}(2^{n+2}\theta_2)=2^{-(n+1)}\cos^{-1}2\theta_2\cos^{-1}4\theta_2\cdots\cos^{-1}(2^{n+1}\theta_2)<2^{-(n+1)}\cos^{-(n+1)}(2^{n+1}\theta_2)<2^{-n}<\frac{\varepsilon}{4}$. Choosing δ small enough to satisfy the above conditions for θ_2 and setting $\theta_2=\theta(s_1)$ we can now observe that $\alpha(\theta)$ reaches $\frac{\pi}{2}-\varepsilon$ for a $\theta=\theta_3<2^{n+1}\theta_2<\varepsilon$. Let $\theta_3=\theta(s_0)$, then $\dot{\alpha}>2\cos\alpha\cot 2\theta>2\sin\varepsilon\cot 2\theta>\varepsilon\cot 2\theta_3>\varepsilon(\cot(2^{n+1}\theta_2))>\frac{\varepsilon}{2}\sin^{-1}\frac{\varepsilon^2}{4}>\frac{2}{\varepsilon}$ for $s_1\leq s\leq s_0$. Since $\alpha(s_0)-\alpha(s_1)<\frac{\pi}{2}$ and $s_0-s_1>s_0-\frac{\varepsilon}{8}$, it follows that $s_0<\varepsilon$. This finishes case (a).

(b) $\alpha\in(\frac{\pi}{2}+\varepsilon,\frac{3\pi}{2})$. From Lemma 1 $\alpha(s)$ has no relative minimum when $\alpha(s)\in(\pi,\frac{3\pi}{2})$. If $\dot{\alpha}(0)\leq 0$, it follows that $\dot{\alpha}(s)<0$ as long as $\alpha(s)\in(\frac{\pi}{2},\frac{3\pi}{2})$ and $\theta(s)<\theta_0$. By a similar argument as in (a) the conclusion follows in this case. If $\dot{\alpha}(0)>0$, there are the following possibilities:

(i) $\alpha(s)$ increases post $\alpha=\frac{3\pi}{2}$; this reduces to (a).

(ii) $\alpha(s)$ increases to $\frac{3\pi}{2}$ and the solution enters the boundary $\theta\equiv 0$.

(iii) $\alpha(s)$ reaches a relative maximum $\alpha_m=\alpha(s_m)\leq\frac{3\pi}{2}$. If $\alpha_m=\frac{3\pi}{2}$, $\dot{\alpha}(s_m)=0$ gives $r(s_m)=\frac{\pi}{2}$, by the uniqueness theorem for differential equations this would be the equator solution, which is a contradiction. Hence $\alpha(s_m)<\frac{3\pi}{2}$, and $\dot{\alpha}(s)<0$ for $s>s_m$; this reduces to the case $\dot{\alpha}(0)\leq 0$. The estimate on s_0 is obtained as above.

Finally, in case $\theta\in(\frac{\pi}{4}-\delta,\frac{\pi}{4})$ the proof proceeds in the same way.

q.e.d.

4. Deformation of the equation and the local structure of the corner singularity.

The last section says nothing about the corner singularities $r = 0,\pi$; for a closer study of these it is advantageous to approximate by a simpler homothetically invariant equation. The extra symmetry of the latter enables one to reduce it to a two-dimensional dynamical system, which is analyzed by the Poincaré-Bendixon method.

Let $k>0$. We define:

$$(*)_k:\quad \begin{aligned} \dot{r} &= \cos\alpha \\ \dot{\theta} &= k\sin\alpha\sin^{-1}kr \\ \dot{\alpha} &= -(2m-1)k\sin\alpha\sin^{-1}kr\cos kr + 2k\cos\alpha\sin^{-1}kr\, K_\theta \end{aligned}$$

in the region $r\in[0,\frac{\pi}{k}]$, $\theta\in[0,\frac{\pi}{4}]$.

For $k = 1$ this coincides with $(*)$. For $k = 0$ we have the limit equation:

$$(*)_0:\quad \begin{aligned} \dot{r} &= \cos\alpha \\ \dot{\theta} &= r^{-1}\sin\alpha \\ \dot{\alpha} &= -(2m-1)r^{-1}\sin\alpha + 2r^{-1}\cos\alpha\, K_\theta \end{aligned}$$

in the region $r>0$, $\theta\in[0,\frac{\pi}{4}]$.

Definition 2.

We denote by $\Gamma_{k:r,\theta,\alpha}(s) = (r_k(s),\theta_k(s),\alpha_k(s))$ the solution of $(*)_k$ with initial conditions (r,θ,α) at $s = 0$, and by $\gamma_{k:r,\theta,\alpha}(s) = (r_k(s),\theta_k(s))$ its projection to orbit space. As in definition 1 $\Gamma_{k:r}$ and $\gamma_{k:r}$ are the special cases of $\theta = 0$, $\alpha = \frac{\pi}{2}$. If $c>0$, we denote by $c\Gamma_{k:r,\theta,\alpha}(s)$ the homothetic image of $\Gamma_{k:r,\theta,\alpha}(s)$, i.e. $c\Gamma_{k:r,\theta,\alpha}(s) = (cr_k(s),\theta_k(s),\alpha_k(s))$.

Proposition 5.

We have $\Gamma_{r,\theta,\alpha}(s) = k\Gamma_{k,rk^{-1},\theta,\alpha}(sk^{-1})$.

Proof. Straighforward differentiation.

From this proposition it follows that solution of $(*)$ can be analyzed by homotheties of solutions of $(*)_k$. For small k, $(*)_k$ is approximated by $(*)_0$, and we now analyze this system.

In the (θ,α)-plane an equivalent system under reparameterization is:

$$(**)_0:\quad \dot\theta = \sin\alpha\,\sin 4\theta$$

$$\dot\alpha = -(2m-1)\sin\alpha\,\sin 4\theta + 4\cos\alpha\, L_\theta,$$

where $L_\theta = (m-2)\cos^2 2\theta - \sin^2 2\theta$.

Singularities of $(**)_0$:

(A): $\theta = \theta_0$, $\alpha = 0,\pi$, corresponding to the solution $\theta\equiv\theta_0$ of $(*)_0$.

(B): $\theta = 0, \frac{\pi}{4}$, $\theta = \pm\frac{\pi}{2}$, corresponding to all solutions of $(*)_0$ with initial values $r\in(0,\pi)$, $\theta = 0,\frac{\pi}{4}$.

Proposition 6

For $m<8$ the sigularity (A) of $(**)_0$ is a focal point. For $m>8$ the singularity (A) is a nodal point, with generic direction of entry $(1,-\frac{1}{2}(2m-1-((2m-1)^2-32(m-1))^{\frac{1}{2}})$ and exceptional direction of entry $(1,-\frac{1}{2}(2m-1+((2m-1)^2-32(m-1))^{\frac{1}{2}})$ in (θ,α)-space. Furthermore, the singularity $(0,\frac{\pi}{2})$ is always a saddle point with separatrices given by the α-axis and by $(m-1,-(2m-1))$, and $(\frac{\pi}{4},-\frac{\pi}{2})$ is always a saddle point with separatrices given by the α-axis and by $(2,-(2m-1))$.

Proof. The matrix of $(**)_0$ at the sigularity (A) is given by $2\sin 2\theta_0 \cos 2\theta_0 \begin{pmatrix} 0 & 1 \\ -8(m-1) & -(2m-1) \end{pmatrix}$ with eigenvalues $\sin 2\theta_0 \cos 2\theta_0(-(2m-1)\pm((2m-1)^2-32(m-1))^{\frac{1}{2}})$, i.e. for $m<8$: two conplex eigenvalues and focal type singularity, for $m>8$: two negative eigenvalues and nodal type singularity. The focal point case $m<8$ is the one investigated in detail by Wu-Yi Hsiang (9, 10). The proof of the proposition is easily completed by computing the eigenvectors at the various singularities.

q.e.d.

From now on we always assume $m>8$.

Proposition 7

The separatrix (other than the α-axis) from $(0,\frac{\pi}{2})$ enters the nodal point $(\theta_0,0)$ along the generic direction of entry, without first crossing $\theta = \theta_0$.

Remark

We believe this result may be known to specialists; for lack of a reference and for completeness we indude the details. The above proposition is equivalent to the statement that the one-parameter family of solution curves $\gamma_{0;r}$, $r>0$, of $(*)_0$ never cross the meridian $\theta\equiv\theta_0$. In (14) Lawson proves that if any such crossing should occurr at a point p, the length of the curve $\theta\equiv\theta_0$ from p to the origin is less than the length of $\gamma_{0;r}$ from p to the initial points $(r,0)$, relative to the modified metric $d\bar{s}^2 = v^2 ds^2$. This result already implies non-interior regularity for the solution of the Plateau problem in R^{2m} with boundary equal to the orbit p (see (14)).

Proof. Here $\dot{\alpha}<0$ initially. For $\theta<\theta_0$ we have $\dot{\alpha}>0$ at $\alpha = 0$ and $\dot{\alpha}<0$ at $\alpha = \frac{\pi}{2}$ (from $(**)_0$), it follows that α remains in $(0,\frac{\pi}{2})$ before any crossing of $\theta = \theta_0$. Let $v = \theta_1^{-1}\alpha+(m-\frac{1}{2})$, where $\theta_1 = \theta-\theta_0$. Here $v>0$ for $-\theta_1>\pi(2m-1)^{-1}$, in particular $v(s)>0$ initially $(\theta_0>\pi(2m-1)^{-1})$.

Lemma 2

$v(s)$ is positive for all s.

Proof. Direct computation and substitution from $(**)_0$ gives:

$$(***)_0\colon\ \dot{v} = -v^2\alpha^{-1}\sin\alpha\sin 4\theta + \frac{(2m-1)^2}{4}\alpha^{-1}\sin\alpha\sin 4\theta + 4L_\theta\theta_1^{-1}\cos\alpha .$$

We now prove that $\dot{v}>0$ at $v = 0$; it then follows that $v(s)$ can never reach 0. Substituting $\theta = \theta_0+\theta_1$ in L_θ, we have: $-\theta_1^{-1}L_\theta = F(\theta_1)\theta_1^{-1}\sin 2\theta_1$, where $F(\theta_1) = -(m-3)\sin 2\theta_1 + 2(m-2)^{\frac{1}{2}}\cos 2\theta_1$. Substituting from Remark 2, (ii), we have: $F(-\theta_0) = (m-1)^{\frac{1}{2}}(m-2)^{\frac{1}{2}}$ and $F(0) = 2(m-2)^{\frac{1}{2}}$. The maximum value of F is $m-1$ at $\sin 2\theta_1 = -(m-1)^{\frac{1}{2}}(m-3)$. We only have to check $v(s)$

when $-\theta_1<\pi(2m-1)^{-1}$, i.e. where $\sin(-2\theta_1)<\sin\frac{\pi}{7}<(m-3)(m-1)^{-\frac{1}{2}}$. In the region $\theta_1\in(0,-(2m-1)^{-1}\pi)$ we have $F(\theta_1)<F(-(2m-1)^{-1}\pi) = (m-3)\sin((2m-1)^{-1}2\pi) + 2(m-2)^{\frac{1}{2}}\cos(2\pi(2m-1)^{-1})<2(m-2)^{\frac{1}{2}}+\pi$. Hence $-\theta_1^{-1}L_\theta\leq\theta_1^{-1}\sin 2\theta_1 + F(\theta_1)<4(m-2)^{\frac{1}{2}}+2\pi$. From $(***)_0$ we have $\dot{v}>0$ at $v = 0$ iff $-16\theta_1^{-1}L_\theta<(2m-1)^2\sin 4\theta\ \alpha^{-1}\sin\alpha\cos^{-1}\alpha$; it is then suficient that $64(m-2)^{\frac{1}{2}} + 32\pi<(2m-1)^2\sin 4\theta$, where $\theta\in(\theta_0-\pi(2m-1)^{-1},\theta_0)$. For $m\geq 9$ we have $\theta_0-\pi(2m-1)^{-1}>0{,}41$ and $4\theta\in(\frac{\pi}{2},4\theta_0)$, $\sin 4\theta>\sin 4\theta_0 = 2(m-2)^{\frac{1}{2}}(m-1)^{-1}$ in the region. Hence it suffices to check that $2(m-2)^{\frac{1}{2}}(m-1)^{-1}>(64(m-2)^{\frac{1}{2}} + 32\pi)(2m-1)^{-2}$. This is quickly checked for $m>11$. For the remaining four values of m it is easy to sharpen the above estimates sufficiently in the relevant region. This concludes the proof of Lemma 2.

Proof of Proposition 6: Since $v(s)$ would approach $-\infty$ when θ crosses θ_0, this would contradict Lemma 2. It now follows that the separatrix must enter the nodal point $(\theta_0,0)$ from above without first encircling it. The proof of Lemma 2 holds for any solution curve $\Gamma_{0;r,\theta,\alpha}(s)$ of $(*)_0$ with $\theta\in(0,\theta_0),\alpha\in(0,\frac{\pi}{2}]$, $v(0)>0$. Consider a one-parameter family $\Gamma_t(s) = (\theta_t(s),\alpha_t(s))$ of solution curves of $(**)_0$ such that $\theta_t(0) = t$, $\alpha_t(0) = c\in(0,\frac{\pi}{2})$, $\gamma_0(0)$ lies on the separatrix. Then $\gamma_t(s)$ crosses $\theta = \theta_0$ for some $t<\theta_0$. Since any such crossing is transversal, this crossing condition is open. Let $t_1 = \sup\{t<\theta_0, \gamma_t(s)$ does not cross $\theta = \theta_0\}$, then $\gamma_{t_1}(s)$ does not cross. By the uniqueness theorem for differential equations γ_{t_1} must be the unique solution curve which enters the nodal point along the exceptional direction.

q.e.d.

5. Some qualitative features of solution curves.

Propositon 8

Let $b\in(0,\frac{\pi}{2})$ and let $\Gamma_b(s) = (r(s),\theta(s),\alpha(s))$ be as in Definition 1. Then there exists a positive s_1 such that $\theta(s_1) = \theta_0$, $r(s)<\frac{\pi}{2}$, $\alpha(s_1)\in(0,\frac{\pi}{2})$, and $\dot{\alpha}(s)<0$ for $s\in(0,s_1]$, i.e. $\gamma_b(s) = (r(s),\theta(s))$ escapes the region III: $(r,\theta)\in(0,\frac{\pi}{2})\times(0,\theta_0)$ by crossing $\theta\equiv\theta_0$.

Proof. In III we have $\dot{\alpha}>0$ at $\alpha = 0$ and $\dot{\alpha}<0$ at $\alpha = \frac{\pi}{2}$ (from $(*)$), it follows that $\alpha(s)\in(0,\frac{\pi}{2})$ until $\gamma_b(s)$ escapes III, and that $\dot{\alpha}(s)<0$ for small s. By Lemma 1, (iii), $\alpha(s)$ has no relative minimum for $\alpha\in(0,\frac{\pi}{2})$, hence $\dot{\alpha}(s)$ remains nagative until the escape from III. By $(*)$ $\dot{\alpha}>0$ at $r = \frac{\pi}{2}$, $\theta<\theta_0$, hence $\gamma_b(s)$ can only escape across $\theta\equiv\theta_0$.

q.e.d.

Let $\Gamma_b(s) = (r_b(s),\theta_b(s),\alpha_b(s))$ be as in Definition 1, and let $R_b = r_b(s_b)$ be the first relative maximum of $r_b(s)$.

Proposition 9

Let $b\in(0,\frac{\pi}{2})$. Then s_b varies continuously with b. All critical points for $\alpha_b(s), s\in[0,s_b]$ are non-degenerate and occur in the interior $(0,s_b)$.

Proof. By theorem 3 $\dot{r}_b(s) = \cos\alpha_b(s)$ varies continuously with b. By Lemma 1, $\ddot{r}_b(s) = (2m-1)\cot r_b(s)$ at a critical point for $r_b(s)$. Hence an inflection point coincides with a critical point at $s = s'$ only if $\cos\alpha_b(s') = 0$, $\cos r_b(s') = 0$, i.e. for the equator solution, $b = \frac{\pi}{2}$. It follows that s_b is continuous for $b \in (0,\frac{\pi}{2})$. We have $\mathring{\alpha}_b(s_b) = -(2m-1)\sin \alpha(s_b)\sin^{-1}R_b \cos R_b \neq 0$ unless $R_b = \frac{\pi}{2}$, $\cos \alpha_b(s_b) = 0$; which again would imply $b = \frac{\pi}{2}$ by the uniqueness theorem for differential equations. Let $s_1 \in (0,s_b)$ be a critical point for $\alpha_b(s)$, from Lemma 1, (iii) we conclude that it is non-degenerate unless $\sin \alpha_b(s_1) = 0$. But then $\mathring{\alpha}(s_1) = 0$ gives $K_\theta(s_1) = 0$, i.e. $\theta(s_1) = \theta_0$. By the uniqueness theorem again this must be the meridian solution $\theta \equiv \theta_0$, i.e. $b = 0$.

q.e.d.

Definition 3

Let $I_1(b)$, $I_2(b)$ and $I(b) = I_1(b)+I_2(b)$ be the number of relative maxima, relative minima and critical points for $\alpha_b(s)$, $s \in (0,s_b]$, respectively, defined if $\theta_b(s_b) \in (0,\frac{\pi}{4})$.

Proposition 10

$I_1(b)$, $I_2(b)$ and $I(b)$ are locally constant as long as $\theta_b(s_b) \in (0,\frac{\pi}{4})$. If $\theta_b(s_b) = 0$, I_1 is constant around b, but I_2 may jump ± 1 at b. Similarly, if $\theta_b(s_b) = \frac{\pi}{4}$, I_2 is constant around b, whereas I_1 may have a jump ± 1.

Proof. Since critical points in $(0,s_b)$ are non-degenerate, they are stable; since no end point can be critical, it follows that $I_1(b)$, $I_2(b)$, $I(b)$ are locally constant. Now, assume $c \in (0,\frac{\pi}{2})$, $\theta_c(s_c) = 0$, $r_c(s_c) > \frac{\pi}{2}$. Then $\alpha_c(s) \to -\frac{\pi}{2}+$ as $s \to s_c-$. For any positive ε there exists a δ such that for $|b-c| < \delta$, we have $\alpha_b(s_c-\delta) \in (-\frac{\pi}{2},-\frac{\pi}{2}+\varepsilon)$, $\theta_b(s_c-\delta) \in (0,\varepsilon)$, $r_b(s_c-\delta) > \frac{\pi}{2}$. If $\mathring{\alpha}_b(s_c-\delta) \geq 0$, it follows from Lemma 1, (iii) that $\mathring{\alpha}_b(s) > 0$ until $\alpha_b = 0$, and from $(*)$ that $\mathring{\alpha}_b(s) > 0$ until $\alpha(s) = \frac{\pi}{2}$, i.e. $s = s_b$. If

$\dot{\alpha}_b(s_c-\delta)<0$, we either have: (i) $\dot{\alpha}_b(s)<0$ until $\alpha_b(s) = -\frac{\pi}{2}$ for $s = s_b$, or (ii) $\alpha_b(s)$ decreases until it reaches a relative minimum at $s = s'$, by Lemma 1, (iii), $\alpha(s')>-\frac{\pi}{2}$, and by (*) and Lemma 1, (iii), $\dot{\alpha}(s)>0$ for $s\in(s',s_b)$. It follows that I_1 is constant and I_2 may jump ± 1 as b crosses c. A similar argument near $\theta = \frac{\pi}{4}$ shows the rest of the proposition.

q.e.d.

Remark

By observing that $\{\gamma_b\}$ is a variation through geodesics relative to the modified metric $d\bar{s}^2 = v^2 ds^2$, and considering the corresponding Jacobi-field along γ_c, it is not hard to see that such jumps do in fact occur.

Theorem 4

Let $0<b_1<b_2<\frac{\pi}{2}$, and assume that $I_1(b_2) = 0$, $I_1(b_1)\geq 1$. Then $\theta_b(s_b) = \frac{\pi}{4}$ for some $b\in(b_1,b_2)$.

This follows from Proposition 10.

Corollary 3

Let b_1 and b_2 be as in Theorem 4. Then there exists a non-equatorial minimal imbedding of S^{2m-1} into $S^{2m}(1)$.

Proof. This follows from Theorem 4 and Theorem 2 applied to the curve $\gamma_b(s) = (r_b(s),\theta_b(s))$ for $s\in[0,s_b]$.

6. Analysis of small perturbations of the equator solution.

Corollary 3 reduces the spherical Bernstein problem to estimating the variation of the number of critical points of $\alpha_b(s)$ along a one-parameter family of pieces of solution curves $\Gamma_b(s)$, $b\in(0,\frac{\pi}{2})$. In this section some analytical effort succeeds in providing suficiently good estimates near the end point $\frac{\pi}{2}$ of the interval.

By the discussion of section 3 we may consider $\Gamma_{\frac{\pi}{2}}(s)$ as defined for all s, with a discontinuity in $\alpha_{\frac{\pi}{2}}(s)$ at $s = k\frac{\pi}{4}$, $k\in\mathbb{Z}$, which disapears when imposing a suitable metric on phase space. A corresponding "continuous dependance" on initial conditions beyond intersections with the singular boundary is provided at $b = \frac{\pi}{2}$ by the following.

Proposition 11

For any n and any $\varepsilon>0$ there exists a $\delta>0$ such that

$|r_b(s)-\frac{\pi}{2}|<\varepsilon$ for $s\in[0,n\frac{\pi}{2}]$.

$|\theta_b(s)-(s-k\frac{\pi}{2})|<\varepsilon$ for $s\in[k\frac{\pi}{2},(k+\frac{1}{2})\frac{\pi}{2}]$, $k = 0,\ldots,n-1$

$|\theta_b(s)-(k\frac{\pi}{2}-s)|<\varepsilon$ for $s\in[(k-\frac{1}{2})\frac{\pi}{2},k\frac{\pi}{2}]$, $k = 1,\ldots,n$

$|\alpha_b(s)-\frac{\pi}{2}|<\varepsilon$ for $s\in[k\frac{\pi}{2},(k+\frac{1}{2})\frac{\pi}{2}-\varepsilon]$, $k = 0,\ldots,n-1$

$|\alpha_b(s)+\frac{\pi}{2}|<\varepsilon$ for $s\in[(k-\frac{1}{2})\frac{\pi}{2}+\varepsilon,k\frac{\pi}{2}-\varepsilon]$, $k = 1,\ldots,n$

whenever $b\in(\frac{\pi}{2}-\delta,\frac{\pi}{2})$.

For convenience we give:

Definition 4

The regions I-IV in orbit space are defined by:

I: $(r,\theta)\in(\frac{\pi}{2},\pi)\times(\theta_0,\frac{\pi}{4})$

II: $(r,\theta)\in(0,\frac{\pi}{2})\times(\theta_0,\frac{\pi}{4})$

III: $(r,\theta)\in(0,\frac{\pi}{2})\times(0,\theta_0)$

IV $(r,\theta)\in(\frac{\pi}{2},\pi)\times(0,\theta_0)$.

We first prove:

Lemma 2

Let $0<\mu<0,1$. Then there exists a $\delta_1>0$ and an $s_2\in[\frac{\pi}{4},\frac{\pi}{4}+\mu]$ such that for $b\in(\frac{\pi}{2}-\delta_1,\frac{\pi}{2})$:

$|r_b(s)-\frac{\pi}{2}|<\mu$ for $s\in[0,s_2]$

$|\theta_b(s)-(\frac{\pi}{2}-s)|<\mu$ for $s\in[\frac{\pi}{4},s_2]$

$|\alpha_b(s)-\frac{\pi}{2}|<\mu$ for $s\in[0,\frac{\pi}{4}-\mu]$, $|\alpha_b(s_2)+\frac{\pi}{2}|<\mu$.

Proof. Let $\mu_2=\frac{1}{4}\mu$. By Proposition 4 we may find a $\mu_1\in(0,\mu_2)$ such that if $\frac{\pi}{4}-\theta_b(s_1)<2\mu_1$, then $|\frac{\pi}{2}+\alpha(s_2)|<\mu_2$ for some $s_2\in[s_1,s_1+\mu_2]$. By Theorem 3 we may find a $\delta>0$ such that $|r_b(s)-\frac{\pi}{2}|<\mu_1$, $|\theta_b(s)-s|<\mu_1$, and $|\alpha_b(s)-\frac{\pi}{2}|<\mu_1$ for $s\in[0,\frac{\pi}{4}-\mu_1]$, when $b\in(\frac{\pi}{2}-\delta,\frac{\pi}{2})$. Setting $s_1=\frac{\pi}{4}-\mu_1$, we have $\frac{\pi}{4}-\theta_b(s_1)=\frac{\pi}{4}-s_1+s_1-\theta_b(s_1)\leq 2\mu_1$. Furthermore, $|r_b(s)-\frac{\pi}{2}|\leq\mu_1+\mu_2<\mu$ for $s\in[s_1,s_2]$, $|\theta_b(s)-s|\leq\mu_1+3\mu_2<\mu$ for $s\in[s_1,\frac{\pi}{4}]$ and $|\theta_b(s)-(\frac{\pi}{2}-s)|<2\mu_1+2\mu_2<\mu$ for $s\in[\frac{\pi}{4},s_2]$ (recall that from (*) we get $|\dot{\theta}_b|<2$ in the region $|r_b-\frac{\pi}{2}|<\mu$).

q.e.d.

To reach the conclusion of the proposition we need to apply continuous dependance on initial conditions. Since $\gamma_b(s_2)$ approaches the singular boundary $\theta=\frac{\pi}{4}$ as $\mu\to 0$ (and hence $b\to\frac{\pi}{2}$), the point $s=s_2$ is useless for obtaining estimates.

Lemma 3

Let $\nu>0$ and let $s_3=\frac{\pi}{4}+\frac{1}{2}(\frac{\pi}{4}-\theta_0)$. Then there exists a $\mu\in(0,\nu)$ and a corresponding s_2 as in Lemma 2 such that $|r_b(s)-\frac{\pi}{2}|<\nu$, $|\theta_b(s)-(\frac{\pi}{2}-s)|<\nu$, and $|\alpha_b(s)+\frac{\pi}{2}|<\nu$ for $s\in[s_2,s_3]$.

Proof. By Lemma 2 it suffices to find a constant K such that $|r_b(s)-\frac{\pi}{2}|$, $|\theta_b(s)-(\frac{\pi}{2}-s)|$, and $|\alpha_b(s)+\frac{\pi}{2}|$ are less than $K\mu$ for $s\in[s_2,s_3]$, i.e. to control these quantities in terms of μ. By Lemma 1 (ii), $\alpha_b(s)\in(-\pi,0)$ for $s\in[s_2,s_3]$.
In region II: By (*) $\dot{\alpha}>0$ at $\alpha=-\frac{\pi}{2}$, hence $\alpha_b(s)\in(-\frac{\pi}{2},0)$ implies $\alpha_b(s+t)\in(-\frac{\pi}{2},0)$ as long as $\gamma_b(s+t)\in II$. By (*) $|r_b(s)-\frac{\pi}{2}|$ must decrease. If $\dot{\alpha}_b(s)<0$, $|\alpha_b(s)+\frac{\pi}{2}|$ decreases, otherwise $\dot{\alpha}_b(s)<-(2m-1)\sin\alpha_b(s)\cot r_b(s)$, hence we control both $|r_b(s)-\frac{\pi}{2}|$ and $|\alpha_b(s)+\frac{\pi}{2}|$ in terms of μ in this region. By $|\theta_b(s_2)-(\frac{\pi}{2}-s_2)|<\mu$ and $\dot{\theta}_b(s)=\sin\alpha_b(s)\sin^{-1}r_b(s)$ it follows directly that we also control $|\theta_b(s)-(\frac{\pi}{2}-s)|$ in terms of μ.
In region I: For $\alpha\in(-\frac{\pi}{2},0)$, $\dot{\alpha}<0$, i.e. $|\alpha_b(s)+\frac{\pi}{2}|$ decreases. Since $\dot{r}_b(s)=\cos\alpha_b(s)$ decreases, we also control $|r_b(s)-\frac{\pi}{2}|$ in terms of μ in this region. For $\alpha\in(-\pi,-\frac{\pi}{2})$, $|r_b(s)-\frac{\pi}{2}|$ decreases. If $\dot{\alpha}_b(s)>0$, $|\alpha_b(s)+\frac{\pi}{2}|$ decreases, otherwise we control $|\alpha_b(s)+\frac{\pi}{2}|$ in terms of μ by applying the estimate $0>\dot{\alpha}_b(s)>-(2m-1)\sin\alpha_b(s)\cot r_b(s)$. Control of $\theta_b(s)$ then follows as above.

Finally, $\gamma_b(s)$ may cross back into region II with $\alpha_b(s) \in (-\pi, -\frac{\pi}{2})$. By $(*)$, $\dot{\alpha}_b > 0$ at $\alpha = -\frac{\pi}{2}$ now, hence $\alpha_b(s) \in (-\pi, -\frac{\pi}{2})$, $\dot{\alpha}_b(s) > 0$ until $\gamma_b(s)$ leaves II again. So $|\alpha_b(s) + \frac{\pi}{2}|$ decreases, and $|r_b(s) - \frac{\pi}{2}|$, $|\theta_b(s) - (\frac{\pi}{2} - s)|$ are controlled in terms of μ as above.

q.e.d.

Proof of Propostion 11. Let $\nu_1 > 0$. By continuous dependance on initial conditions at the point $\Gamma_{\frac{\pi}{2}}(s_3) = (\frac{\pi}{2}, \frac{\pi}{8} + \frac{\theta_0}{2}, -\frac{\pi}{2})$ there exists a $\nu \in (0, \nu_1)$ such that for $|r_b(s_3) - \frac{\pi}{2}|$, $|\theta_b(s_3) - (\frac{\pi}{2} - s_3)|$, and $|\alpha_b(s_3) + \frac{\pi}{2}| < \nu$ we have $|r_b(s) - \frac{\pi}{2}|$, $|\theta_b(s) - (\frac{\pi}{2} - s)|$, and $|\alpha_b(s) + \frac{\pi}{2}| < \nu_1$ for $s \in [s_3, \frac{\pi}{2} - \nu_1]$. Determine μ as in Lemma 3 and δ_1 as in Lemma 2. Let $n = 1$. We have now solved the problem for $s \in [0, \frac{\pi}{2} - \nu_1]$. By the same argument as in Lemma 2 (near $\theta = 0$) we can extend beyond $s = \frac{\pi}{2}$. Repetition of this argument finishes the proof for general n.

q.e.d.

Theorem 5

There exists a $\delta > 0$ such that $I_1(b) = 0$ for $b \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$.

Proof. Let $b \in (0, \frac{\pi}{2})$. Then $\gamma_b(s)$ starts out in III, by Proposition 8 it crosses into II at $s = s_1$. Let $0 < \varepsilon < 0,1$, choose δ as in Proposition 11, and $b \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$. By Proposition 8 and $(*)$, $\dot{\alpha}_b(s) < 0$ for $s \in [0, \frac{\pi}{4} - \varepsilon]$. By Lemma 1, (i) $\alpha_b(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ until either: (a) $\gamma_b(s)$ crosses into I for $s = s_2$, or (b) $\gamma_b(s)$ crosses back into III for $s = s_2$. By $(*)$ and Lemma 1, (iii), $\alpha_b(s)$ has no relative maximum for $s \in [0, s_2]$.

Case (a): Here $\dot{\alpha}_b(s_2)<0$. By Lemma 1, (iii) and (*) $\dot{\alpha}_b(s_2+t)<0$ until either: $\alpha_b(s) = -\frac{\pi}{2}$ at $s = s_b$, in which case $I_1(b) = 0$, or: $\gamma_b(s)$ crosses into IV at $s = s_3$. Then either $\dot{\alpha}_b(s_3+t)<0$ until $\alpha_b(s) = -\frac{\pi}{2}$ at $s = s_b$ as above, or $\alpha_b(s)$ has a relative minimum at $s = s_4$, with $\dot{\alpha}_b(s_4+t)>0$ as long as $\gamma_b(s_4+t)$ remains in IV, and $t\in(0,s_b-s_4)$. If $s_b<\frac{\pi}{2}+\varepsilon$, it now follows that $I_1(b) = 0$. Otherwise we reach the following conclusion:

(T): $\gamma_b(\frac{\pi}{2}+\varepsilon)\in IV$ and $\alpha_b(s)$ has no relative maximum for $s\in(0,\frac{\pi}{2}+\varepsilon)$

Case (b): In this case $\dot{\alpha}_b(s_2)>0$, $r_b(s_2)\in(\frac{\pi}{2}-\varepsilon,\frac{\pi}{2})$, $\theta_b(s_2) = \theta_0$; then $\alpha_b(s)\in(-\frac{\pi}{2},-\frac{\pi}{2}+\varepsilon)$ and $\dot{\alpha}_b(s)>0$ for $s\in[s_2,\frac{\pi}{2}-\varepsilon]$, as long as $\gamma_b(s)$ stays in III.

Lemma 4

By choosing ε sufficiently small, we always have that $r_b(s) = \frac{\pi}{2}$ for an $s\in[s_2,\frac{\pi}{2}-\varepsilon]$.

Proof. This certainly holds in case (a). We control the intersections of $\gamma_b(s)$ with the equator solution by estimating the auxilary function $u(s) = \cos\alpha_b(s)\cos^{-1}r_b(s)$ along the orbit. Differentiation and substitution from (*) gives:
$\dot{u} = (2m-1)\sin^2\alpha\sin^{-1}r-2K_\theta u\sin\alpha\sin^{-1}r+u^2\sin r$. Then $\dot{u}>(1-\varepsilon)[(2m-1)+u^2]$ for $s\in[s_2,\frac{\pi}{2}-\varepsilon]$ in case (b). Comparing with the solution $u = \sqrt{2m-1}\tan((1-\varepsilon)\sqrt{2m-1}\,s+c)$ of the equation $\dot{u} = (1-\varepsilon)[(2m-1)+u^2]$ with initial condition $u(s_2) = \cos\alpha_b(s_2)\cos^{-1}r_b(s_2)>0$, we observe that $u(s)$ increases to infinity before $s = s_2+\frac{\pi}{2}(2m-1)^{-\frac{1}{2}}(1-\varepsilon)^{-1}<\frac{\pi}{2}-\theta_0+\varepsilon+\frac{\pi}{2}(2m-1)^{-\frac{1}{2}}(1-\varepsilon)^{-1}$. Recalling that $\theta_0 = \mathrm{Arctan}\sqrt{m-2}$ and $m\geq 8$, the conclusion of Lemma 4 now follows.

We return to the proof of Theorem 4. Choosing ε small and applying Lemma 4, we see that $\gamma_b(s)$ goes as the last alternative discussed under case (a); i.e. the conclusion (T) holds also in this case. It remains only to discuss $\gamma_b(\frac{\pi}{2}+\varepsilon+t)$ for the case $r_b(\frac{\pi}{2}+\varepsilon)\in(\frac{\pi}{2},\frac{\pi}{2}+\varepsilon)$, $\theta_b(\frac{\pi}{2}+\varepsilon)\in(0,\varepsilon)$, $\alpha_b(\frac{\pi}{2}+\varepsilon)\in(\frac{\pi}{2}-\varepsilon,\frac{\pi}{2})$. By (*) $\dot{\alpha}_b(s)>0$ an $\alpha_b(s)\in(\frac{\pi}{2}-\varepsilon,\frac{\pi}{2})$ for $s\in[\frac{\pi}{2}+\varepsilon,s_b]$ as long as γ_b is in IV, (it is clear that $\gamma_b(s)$ cannot cross back into III for $s<s_b$ (Lemma 1, (i))). Hence $\dot{u}>(1-\varepsilon)[(2m-1)+u^2]$ in this region; by a comparison as in the proof of Lemma 4 it follows that u must increase from $u(\frac{\pi}{2}+\varepsilon)<0$ to $u(s)=0$ for an $s=s_3$ with $\gamma_b(s_3)\in I$. Then $\alpha_b(s_3)=\frac{\pi}{2}$, i.e. $s_3=s_b$, and $\alpha_b(s)$ has no relative maximum for $s\in(0,s_b]$.

q.e.d.

7. Analysis of small perturbations of the meridian solution.

We need the following extension of Theorem 3:

Proposition 12

Let $b\in(0,\frac{\pi}{k})$. Then there exists a unique solution curve $\Gamma_{k;b}$ of $(*)_k$; for this curve $\alpha_{k;b}(0)=\frac{\pi}{2}$ (see Definition 2). The curve $\Gamma_{k,b}(s)$ is analytic in (k,b,s), including $k=0$, as long as s is restricted to an interval where the curve does not hit the singular boundary.

Proof. We need only observe that when reformulating the equations as in the proof of Theorem 3 and expanding the right hand side in power series, the coefficients $a_{\ell qn\nu}$ are analytic in k and b, including $k=0$. From the recursion formula for the coeficients of the solution (11) it follows that the solution is analytic also in k.

q.e.d.

Let $v = \alpha(\theta-\theta_0)^{-1}+(m-\frac{1}{2})\cos r$. For any solution curve $(r,(s),\theta(s),\alpha(s))$ of $(*)$ we define $v(s)$ as v evaluated along that solution curve.

Proposition 13

For any $\varepsilon>0$ there exists a $\delta>0$ and an $s_0\in(0,\varepsilon)$ with $0<\theta_0-\theta_b(s_0)<\varepsilon$, $0<\alpha_b(s_0)<\varepsilon$, $v_b(s_0)>0$, and $0<\theta_0-\theta_b(s)$ for $s\in[0,s_0]$ whenever $b\in(0,\delta)$.

Proof. By Proposition 6, interpreted back to $(*)_0$, we can find a K such that $\Gamma_{0,1}(s) = (r_{01}(s),\theta_{01}(s),\alpha_{01}(s))$ satisfies: $\theta_0-\theta_{01}(s)$, $\alpha_{01}(s)\in(0,\frac{\varepsilon}{2})$, and $\alpha_{01}(s)(\theta_{01}(s)-\theta_0)^{-1}+(m-\frac{1}{2})>\frac{1}{4}((2m-1)^2-32(m-1))^{\frac{1}{2}}$ for $s\geq K$ (recall that the solution enters the singularity along the generic direction). Let R be a tubular neighbourhood of $\{\Gamma_{01}(s)\,|\,s\in[0,K]\}$ of radius less that $\frac{\varepsilon}{2}$ and so small that it does not intersect $\theta\equiv\theta_0$. By Proposition 11 we can find a $\mu>0$ such that $\Gamma_{k,1}(s)$ is in R for $s\in[0,K]$ if $0\leq k<\mu$. Let $\delta = \min(\mu,\varepsilon K^{-1})$, and let $k\in(0,\delta)$. By Proposition 5 with $r = k$ we have $\theta_0-\theta_k(kK)$, $\alpha_k(kK)\in(0,\varepsilon)$ for $k\in[0,\delta)$. Furthermore, by again applying Propositions 11, 5 and 6 it is clear that we can further reduce δ to obtain $v_k(kK) = \alpha_{k1}(K)(\theta_{k1}(K)-\theta_0)^{-1}+(m-\frac{1}{2})\cos r_{k1}(K)>0$.

q.e.d.

We define $z = -\cos r$; then $v = \alpha(\theta-\theta_0)^{-1}-(m-\frac{1}{2})z$.

Lemma 5

For any $\varepsilon>0$ there exists a $\delta>0$ and a $K\in(1-\varepsilon,1+\varepsilon)$ such that:

$$(**)\quad \left|\frac{dv}{dz} - K(1-z^2)^{-1}\left(-v^2+\frac{4m^2-1}{4}z^2-\frac{18m-17}{2}\right)\right|<\varepsilon \quad \text{whenever}$$

$$|\alpha|,\ |\theta-\theta_0|<\delta.$$

Proof. Let $u=\alpha(\theta-\theta_0)^{-1}$. Then:
$\dot{u}=\sin^{-1}r(-\alpha^{-1}(\sin\alpha)v^2+\alpha^{-1}(\sin\alpha)\frac{(2m-1)^2}{4}\cos^2 r +$
$2(\cos\alpha)(\theta-\theta_0)^{-1}K_\theta)$.
$\frac{dv}{dz}=\sin^{-1}r\cos^{-1}\alpha\,\dot{v}=(1-z^2)^{-1}(-v^2\alpha^{-1}\tan\alpha+\frac{(2m-1)^2}{4}z^2\alpha^{-1}\tan\alpha +$
$2K_\theta(\theta-\theta_0)^{-1}-(m-\frac{1}{2})(1-z^2))$. Expanding K_θ we find $\lim_{\theta\to\theta_0}K_\theta(\theta-\theta_0)^{-1}=$
$-4(m-1)$, and the conclusion follows easily.

Definition 5

We define $c_k=(\frac{k}{8})^{\frac{1}{2}}(18m-17)^{\frac{1}{2}}(4m^2-1)^{-\frac{1}{2}}$.

Let $\gamma_b(s)$ have its i-th crossing with $\theta\equiv\theta_0$ at $s=s_i$.

Lemma 6

There exists a $\delta>0$ such that $z(s_1)=-\cos r_b(s_1)>-c_{16}$ whenever $b\in(0,\delta)$.

Proof. Clearly $v(s_1-)=-\infty$. From (**) we see that for any $\varepsilon>0$, there exists a $\delta>0$ such that $\frac{dv}{dz}>(1-\varepsilon)(1-z^{-2})^{-1}(\frac{4m^2-1}{4}z^2-\frac{18m-17}{2})-\varepsilon$ at $v=0$ whenever $b\in(0,\delta)$. Let $\mu>0$. By choosing ε sufficiently small we see that $\frac{dv}{dz}>0$ at $v=0$ for $z>-c_{16}-\mu$, hence v cannot decrease below zero in that region. By (**) $\frac{dv}{dz}$ is bounded in $[-c_{16}-\mu,-c_{16}]$, the conclusion follows by choosing μ sufficiently small.

Proposition 14

For any $\varepsilon>0$ there exists a $\delta>0$ such that $|\theta_0-\theta_b(s)|<\varepsilon$, $|\alpha_b(s)|<\varepsilon$ for $s\in[\varepsilon,\pi-\varepsilon]$ whenever $b\in(0,\delta)$.

Proof. Choose a $\delta>0$ which simultaneously satisfies Proposition 13 and Lemma 6. Then, from Proposition 8 and $(*)$ it follows that $\alpha_b(s)$ and $\theta_0-\theta_b(s)$ are both in $(0,\varepsilon)$ for $s\in[s_0,s_1]$. Hence we obtain an approximation of $\gamma_b(s)$ to the solution $\theta\equiv\theta_0$ with arbitrary accuracy at a point independent of δ, e.g. at $\cos r = \frac{1}{2}c_{16}$, $\theta = \theta_0$, $\alpha = 0$. The conclusion follows by continuous dependance on initial conditions applied to the meridian solution $\theta\equiv\theta_0$ at the above point.

q.e.d.

Corollary 4

For any $\varepsilon>0$ there exists a $\delta>0$ such that $(**)$ holds for the solution curve $\gamma_b(s)$, $s\in[\varepsilon,\pi-\varepsilon]$, whenever $b\in(0,\delta)$.

Proposition 15

There exists a $\delta>0$ such that $s_b>s_4$ for $b\in(0,\delta)$.

This will be proved by estimating the variation of v along γ_b in the interval $z\in[-c_{16},c_{16}]$. Notice that for $s_i<s_b$ we have $v(s_i-) = -\infty$, $v(s_i+) = \infty$, regardless of whether γ_b crosses $\theta\equiv\theta_0$ from above or below. Hence $v(s)$ passes from ∞ to $-\infty$ for $s\in[s_i,s_{i+1}]$, and it suffices to show that this occurs at least four times for $z\in[-c_{16},c_{16}]$.

Consider the equation

$$(***)\quad \frac{dv}{dz} = -v^2 + \frac{4m^2-1}{4}z^2 - \frac{18m-17}{2}.$$

Then $\frac{dv}{dz}<0$ for $z\in(-c_{16},c_{16})$. Let $v(z)$ be the solution of (***) such that $v(-c_{16}+) = -\infty$. When $v(z_1-) = -\infty$ we can extend the solution beyond the discontinuity $z = z_1$ by setting $v(z_1+) = \infty$, etc. Notice that $w(z) = \arctan v(z)$ satisfies the differential equation: $\frac{dw}{dz} = -1+\cos^2 w(1+ \frac{4m^2-1}{4} z^2 - \frac{18m-17}{2})$. By Corollary 4 and comparison it now suffices to show:

Lemma 7

For $x(z) = \arctan v(z)$ we have: $w(-c_{16})-w(c_{16})>4\pi$, i.e. $v(z)$ decreases from ∞ to $-\infty$ at least 4 times in $[c_{-16},c_{16}]$.

The computational details of Lemma 7 are given in the Appendix, and we are then ready to conclude with out main results.

Theorem 6

There exists a $\delta>0$ such that $I_1(b)\geq 1$ for $b\in(0,\delta)$.

Proof. We have $\dot{\alpha}_b(s_1)<0$, furthermore, it follows from (*) that $\dot{\alpha}_b(s_i)$ and $\dot{\alpha}_b(s_{i+1})$ have opposite signs if $r_b(s_i)$, $r_b(s_{i+1})$ are either both in $(0,\frac{\pi}{2})$ or both in $(\frac{\pi}{2},\pi)$, $i = 1,2,3$. By Proposition 14 $\alpha_b(s)$ now has at least one relative maximum for $s\in(0,s_b)$, and the conclusion follows.

q.e.d.

Theorem 7 (Main theorem)

Let $S^{2m}(1)$ be the standard Euclidean sphere of dimension $2m$. Then there exists a minimally imbedded $(2m-1)$-sphere which is different from the equator.

Proof. This now follows form Theorems 5, 6 and Corollary 3.

Theorem 8

An isolated singularity of a minimal hypersurface of an odd dimensional Euclidean space R^{2m+1} cannot in general be detected by its local topological structure.

Proof. The cone to the origin on the example of Theorem 7 is a minimal cone in R^{2m+1} with the vertex as a singular point. Obviously the intersection of this with a sphere around the vertex is topologically a $(2m-1)$-sphere, i.e. topologically indistinguishable from the corresponding intersection around a regular point of the hypersurface.

q.e.d.

Remark

$S(U(2)\times U(m))$ acts on $S^{4m}(1) \subseteq R^{4m+1} \simeq C^2 \otimes C^m \oplus R$ by the representation $\mu_2 \otimes \mu_m \oplus 1$ (μ_k = standard representation of $U(k)$ on C^k). The orbit space has the same parameterization as in the orthogonal case, and the volume functional is given by $v(x_1,x_2,x_3) = cx_1^{2m-3}x_2^{2m-3}(x_1-x_2)^2(x_1+x_2)^2 = \sin^{4m-2} r \sin^{2m-3} 2\theta \cos^2 2\theta$. The modification in the basic equation (*) is now given by:
$\dot{\alpha} = -(4m-2)\sin\alpha \sin^{-1} r \cos r + 2\cos\alpha \sin^{-1} r((2m-3)\cot 2\theta - 2\tan 2\theta)$.
With $z = -\cos r$ and $v = \alpha(\theta-\theta_0)^{-1}(2m-\frac{1}{2})z$ we now have the control equation $\frac{dv}{dz} = (1-z^2)^{-1}(-v^2 + \frac{16m^2-1}{4} z^2 - \frac{36m-17}{2})$; this is the same as (***) when substituting $2m$ for m. Hence the same proof, with the computation in the Appendix, gives the existence also of an $S(U(2)\times U(m))$-invariant minimally imbedded hypersphere in $S^{4m}(1)$. $Sp(2)\times Sp(m)$ acts on $S^{8m}(1) \subseteq R^{8m+1} \simeq H^2 \otimes H^m \oplus R$ by

$\nu_2 \otimes \nu_m \oplus 1$, in this case the volume functional is: $v(x_1,x_2,x_3) = c x_1^{4m-5} x_2^{4m-5} (x_1-x_2)^4 (x_1+x_2)^4 = \sin^{8m-2} r \sin^{4m-5} 2\theta \cos^4 2\theta$. Here $\dot{\alpha} = -(8m-1)\sin\alpha \cos r \sin^{-1} r + 2\cos\alpha \sin^{-1} r((4m-5)\cot 2\theta - 4\tan 2\theta)$, and with $v = \alpha(\theta-\theta_0)^{-1} - (4m-\frac{1}{2})z$ we have the controlling limit equation $\frac{dv}{dz} = (1-z^2)^{-1}(-v^2 + \frac{64m^2-1}{4} - \frac{72m-17}{2})$, which again is (***) when substituting $4m$ for m. Hence we also obtain an $Sp(2) \times Sp(m)$-invariant mininally imbedded hypersphere in $S^{8(m)}(1)$.

Appendix

To prove Lemma 7 we need to estimate v along a sufficiently fine subdivision of $[-c_{16}, c_{16}]$. Let $I_{k,\ell} = [-c_\ell, -c_k]$ and $J_{k,\ell} = [c_k, c_\ell]$ for $0 \leq k < \ell < 16$. Then, for any solution $v(z)$ of (***), we have: $\frac{dv}{dz} < -(v^2 + \frac{18m-17}{32}(16-\ell)$ for $z \in I_{k,\ell}$ or $J_{k,\ell}$. The solution of the equation

$$(****)\ \frac{dv}{dz} = -(v^2 + \frac{18m-17}{32}(16-\ell))$$

is $v = -T_\ell \tan \phi_\ell$, where $T_\ell = 32^{-\frac{1}{2}}(18m-17)^{\frac{1}{2}}(16-\ell)^{\frac{1}{2}}$, $\phi_\ell = T_\ell z + C$ with C a constant. We call ϕ_ℓ the ℓ-phase of v. The increase in ℓ-phase of v over $I_{k,\ell}$ or $J_{k,\ell}$ is estimated by:

$$(1)\quad \Delta\phi_\ell > \frac{63}{128}(16-\ell)^{\frac{1}{2}}(\ell^{\frac{1}{2}} - k^{\frac{1}{2}}).$$

This follows since $\Delta\phi_\ell = T_\ell \Delta z = \frac{1}{16}(18m-17)(4m^2-1)^{-\frac{1}{2}}(16-\ell)^{\frac{1}{2}}(\ell^{\frac{1}{2}}-k^{\frac{1}{2}}) > \frac{63}{128}(16-\ell)^{\frac{1}{2}}(\ell^{\frac{1}{2}}-k^{\frac{1}{2}})$ (since $m \geq 8$).

The amplitude T_ℓ varies for different subintervals, hence we also need to consider a "phase-shift" at end-points:

At $z = -c_k$: $-T_\ell \tan \phi_\ell(-c_k) = -T_k \tan \phi_k(-c_k)$, i.e.

$$(2)\quad \phi_k(-c_k) = \operatorname{Arctan}((16-\ell)^{\frac{1}{2}}(16-k)^{-\frac{1}{2}} \tan \phi_\ell(-c_k)).$$

Similarly, at $z = c_k$:

$$(3)\quad \phi_\ell(c_\ell) = \operatorname{Arctan}((16-k)^{\frac{1}{2}}(16-\ell)^{-\frac{1}{2}} \tan \phi_k(c_k))$$

For the case $\ell = -16$ we compare with the solution $v = (z+c_{16})^{-1}$ of $\frac{dv}{dz} = -v^2$. Then $v(-c_k) < (c_{16}-c_k)^{-1} = 2^{3/2}(4m^2-1)^{\frac{1}{2}}(18m-17)^{-\frac{1}{2}}(4-k^{\frac{1}{2}})^{-\frac{1}{2}}$.

For the k-phase at $z = -c_k$ we then have: $\phi_k(-c_k) = -\text{Arctan}(T_k^{-1} v(-c_k)) > -\text{Arctan}\ \frac{128}{63}\ (4-k^{\frac{1}{2}})^{-\frac{1}{2}}(16-k)^{-\frac{1}{2}}$ $(m \geq 8)$.

Let $k = 15,5$. Then $\phi_k(-c_k) > -1.5488754$. By (1) applied to $\ell = 15,5$, $k = 15$: $\Delta\phi > 0.022281$, hence $\phi_{15,5}(-c_{15}) > -1.526594$.

At each step we now apply the shift formula (2) and then (1):

$\ell = 15$, $k = 14$: $\phi_{15}(-c_{14}) > \text{Arctan}((\frac{16-15,5}{16-15})^{\frac{1}{2}} \tan \phi_{15,5}(-c_{15})) + \frac{63}{128} \cdot 1^{\frac{1}{2}}(15^{\frac{1}{2}}-14^{\frac{1}{2}}) > -1.443689$

$\ell = 14$, $k = 13$: $\phi_{14}(-c_{13}) > \text{Arctan}((\frac{1}{2})^{\frac{1}{2}} \tan \phi_{15}(-c_{14})) + \frac{63}{128}\ 2^{\frac{1}{2}}(14^{\frac{1}{2}}-13^{\frac{1}{2}}) > -1.297258$

For the convenience of the reader we list the further estimates obtained:

$\phi_{13}(-c_{12}) > -1.119225$, $\phi_{12}(-c_{11}) > -0.9151182$, $\phi_{11}(-c_{10}) > -0.6906602$,
$\phi_{10}(-c_9) > -0.4506929$, $\phi_9(-c_8) > -0.1977776$, $\phi_8(-c_7) > 0.0690008$,
$\phi_7(-c_6) > 0.3548584$, $\phi_6(-c_5) > 0.670213$, $\phi_5(-c_4) > 1.0325066$,
$\phi_4(-c_3) > 1.470002$, $\phi_3(-c_{2,798}) > 1.571 > \frac{\pi}{2}$.

Hence it follows that $v(z)$ has reached $-\infty$ (and the first crossing of γ_b with $\theta \equiv \theta_0$ has occurred) before $z = -c_{2,798}$. At the crossing v jumps from $-\infty$ to ∞, and we may continue by comparing with the solution curve of (****) with phase $-\frac{\pi}{2}$ at $z = -c_{2,798}$.

$\ell = 2,798$, $k = 2$: $\phi_{2,798}(-c_2) > -\frac{\pi}{2} + \frac{63}{128}\ (16-2.798)^{\frac{1}{2}}(2,798^{\frac{1}{2}}-2^{\frac{1}{2}}) > -1,108495$

$\ell = 2$, $k = 1.5$: $\phi_2(-c_{1.5}) > \text{Arctan}((\frac{13.202}{14})^{\frac{1}{2}} \tan \phi_{2,798}(-c_2) + \frac{63}{128}\ 14^{\frac{1}{2}}(2^{\frac{1}{2}}-1.5^{\frac{1}{2}}) > -0.747753$

$\ell = 1.5$, $k = 1$: $\phi_{1.5}(-c_1) > -0,317796$

$\ell = 1$, $k = 0.0001$: $\phi_1(-c_{0,0001}) > 1.574 > \frac{\pi}{2}$.

Hence $v(z)$ has again reached $-\infty$ (and γ_b has crossed $\theta \equiv \theta_0$ again) before $z = -c_{0,0001}$. We continue by comparing with the solution of (****) with phase $-\frac{\pi}{2}$ at $z = -c_{0,0001}$.

$\ell = 0{,}0001$, $k = 0$: $\Delta\phi_\ell = 15{,}9999^{\frac{1}{2}} \cdot 0{,}01 \cdot \frac{63}{128}$ over both $I_{k,\ell}$ and $J_{k,\ell}$. Hence $\phi_{0,0001}(c_{0,0001}) > -\frac{\pi}{2} + 2 \cdot \Delta\phi_\ell > -1{,}531422$. We have now reached the region $z>0$. We continue as before, but now use (3) and (1) at each stage.

$k = 0{,}0001$, $\ell = 0{,}5$: $\phi_{0,5}(c_{0,5}) > \mathrm{Arctan}((\frac{15{,}9999}{15{,}5})^{\frac{1}{2}}(-\tan 1.531422)) + \frac{63}{128}(15{,}5)^{\frac{1}{2}}(0{,}5^{\frac{1}{2}} - 0{,}01) > -0{,}181227$.

$\phi_1(c_1) > 0{,}374166$, $\phi_2(c_2) > 1{,}148864$, $\phi_{2,75}(c_{2,75}) > 1{,}58 > \frac{\pi}{2}$. $v(z)$ has decreased to $-\infty$ (and the third crossing with $\theta \equiv \theta_0$ has occurred) before $z = c_{2,75}$. We now compare with the solution of (****) with phase $-\frac{\pi}{2}$ at $z = c_{2,75}$.

$\phi_3(c_3) > -\frac{\pi}{2} + \frac{63}{128}(13^{\frac{1}{2}}(3^{\frac{1}{2}} - 2{,}75^{\frac{1}{2}}) > -1.439940$.

$\phi_4(c_4) > -0{,}988170$, $\phi_5(c_5) > -0{,}622626$, $\phi_6(c_6) > -0{,}313191$,
$\phi_7(c_7) > -0{,}039172$, $\phi_8(c_8) > 0{,}212759$, $\phi_9(c_9) > 0{,}450388$,
$\phi_{10}(c_{10}) > 0{,}676952$, $\phi_{11}(c_{11}) > 0{,}891711$, $\phi_{12}(c_{12}) > 1{,}090671$,
$\phi_{13}(c_{13}) > 1{,}267694$, $\phi_{14}(c_{14}) > 1{,}415526$, $\phi_{15}(c_{15}) > 1{,}525198$,
$\phi_{15,5}(c_{15,5}) > 1{,}560823$, $\phi_{15,8}(c_{15,8}) > 1{,}572 > \frac{\pi}{2}$.

q.e.d.

Thus we have proved - in the nick of time - that the fourth crossing with the meridian solution occurs for $z<c_{16}$. Note that although the equation (***) depends on m, the estimates (1), (2) and (3) do not, this enables us to prove the result uniformly for all $m \geqslant 8$. For special values of m the result can be checked by numerical integration on a computer; to avoid infinities, it is then more practical to integrate the equation of $w(z) = \arctan v(z)$ (section 7). Note that v changes rapidly for z near 0, a program with subintervals $I_{k,\ell}$ of equal size requires far more subintervals (ca. 1000) than our computation.

Remark

Since $\frac{dv}{dz} > -(v^2 + \frac{18m-17}{32}(16-k))$ for $z \in I_{k,\ell}$ or $J_{k,\ell}$, one may carry through an estimate of the absolute change of v, $|\Delta v|$, from above in a similar manner. It is then possible to sharpen Theorem 1 to $I_1(b) = 1$ for $b \in (0,\delta)$, thus indicating that one can only expect one example invariant under $SO(2) \times SO(m)$.

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